RUNS OF INTEGERS WITH EQUALLY MANY DISTINCT PRIME FACTORS

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In memory of Jean Kimberley (1921–2011) and John Kimberley (1945–2011).

Abstract. We survey and update data on runs of consecutive integers each having exactly r distinct prime factors (briefly, of principal rank r). For $3 \le r \le 64$ and other sporadic values, lower bounds are given for the size of longest runs of consecutive integers of principal rank r, together with upper bounds on the first occurrence of such runs. We also prove that there are infinitely many pairs of consecutive integers of principal rank r, for each $r \ge 3$.

1. Introduction

For any positive integer n, the standard notation $\omega(n)$ denotes the number of distinct prime factors of n, as with $\omega(8) = \omega(9) = 1$. The sequence A1221 in OEIS, the Online Encyclopedia of Integer Sequences [14], tabulates an initial segment of the sequence $\omega(\mathbb{Z}^+)$. For brevity, and convenience in discussion, we call $\omega(n)$ the *principal rank* of n, often omitting "principal". In the same spirit, when referring here to positive integers we shall omit "positive" for brevity.

The local structure of $\omega(\mathbb{Z}^+)$ has been the subject of various studies, such as recent work by De Koninck et al [12] on fully heterogeneous blocks. Here, in contrast, we study homogeneous blocks. Schlage-Puchta [15] recently showed that $\omega(n) = \omega(n+1)$ has infinitely many integer solutions, so $\omega(\mathbb{Z}^+)$ contains infinitely many homogeneous blocks of size at least 2.

For any rank $r \ge 1$, the existence of runs of two or more consecutive integers within the constant rank set $P_r = \{n \in \mathbb{Z}^+ : \omega(n) = r\}$ is of considerable interest. We write $n^{[s]} \subset P_r$ when $\{n + i : 0 \le i < s\} \subset P_r$ and $\{n-1, n+s\} \cap P_r = \emptyset$, that is, $n^{[s]}$ denotes a *maximal* run of s

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consecutive integers, beginning with n, all of which have the same rank r. For example, $31^{[2]} \subset P_1$ and $33^{[3]} \subset P_2$. Maximal runs in the principal rank sets P_r were studied in [5].

As remarked in [5], from Mihăilescu [13] we know that $2^{[4]}$ and $7^{[3]}$ are the unique maximal runs of size 4 and 3 in P_1. Each subsequent nontrivial maximal run in P_1 has size 2, and must contain a Fermat or Mersenne prime, so P_1 has infinitely many nontrivial maximal runs if and only if there are infinitely many Fermat or Mersenne primes.

Let $s(P_r)$ denote the maximum size achieved by runs in P_r. It is easy to see that $s(P_r)$ must be finite. If $N_r = p_1 p_2 \cdots p_r$ is the product of the first r primes, it is clear that $s(P_r) < N_{r+1}$. Moreover, there can be only finitely many runs of size $s \geq 2N_r$, in view of Størmer's Theorem [17]. However, the only exact value of $s(P_r)$ known is the obvious $s(P_1) = 4$.

In [5] it was proved that $8 \leq s(P_2) \leq 9$. Further, it was conjectured that $s(P_2) = 8$, in view of the facts that $141^{[8]}$, $212^{[8]}$, $323^{[7]}$ and $2302^{[7]}$ are maximal runs in P_2, and these are the only runs of size greater than 6 in $\{n \in P_2 : n \leq 10^{25}\}$. Indeed, Schneider [16] has since computed that there is no run of size 9 in $\{n \in P_2 : n \leq 10^{700}\}$. For higher ranks, [5] showed that $s(P_3) \geq 16$, $s(P_4) \geq 14$, $s(P_5) \geq 7$, $s(P_6) \geq 3$, and $s(P_r) \geq 2$ for $7 \leq r \leq 10$.

2. Computations for $n \leq 10^{12}$ and $n \leq 10^{13}$

To obtain further concrete information about $\omega(\mathbb{Z}^+)$, we subsequently used Magma [1] to compute the initial segment $\omega(n \leq 10^{12})$. Table 1 summarizes the frequencies of principal ranks for $n \leq 10^{12}$ (see also [11]).

Table 1. Frequencies of principal ranks up to 10^{12}.

r	Frequency
0	1
1	37 607 992 088
2	163 437 431 298
3	293 574 084 591
4	283 019 940 982
5	158 910 601 699
6	52 588 537 590
7	9 868 001 325
8	953 640 790
9	39 306 280
10	462 922
11	434

We made a census of various profiles [2] within $\omega(n \leq 10^{12})$, especially the *plateaux* (maximal nontrivial runs of consecutive integers in one constant rank set P_r). For $r \geq 1$ and $s \geq 2$, let $n(r, s)$ denote the starter of the first plateau of height r and size s in $\omega(\mathbb{Z}^+)$; for example, $n(2, 8) = 141$ and $n(2, 7) = 323$. Note that although the size 8 run at 141 includes two runs of size 7, neither is maximal; the run starting at 323 is the first maximal run of size 7. The plateaux in $\omega(n \leq 10^{12})$ exhibit 63 distinct pairs of parameters (r, s). In particular, $n(4, 23) = 585\,927\,201\,062$ is the starter of the unique largest plateau in $\omega(n \leq 10^{12})$. Every other plateau in $\omega(n \leq 10^{12})$ has size $s \leq 19$. These plateaux yield the improved lower bounds $s(P_3) \geq 18$, $s(P_4) \geq 23$, $s(P_5) \geq 10$, $s(P_6) \geq 6$ and $s(P_7) \geq 3$.

For fixed rank r, the sequence $(n(r, s) : s \geq 2)$ corresponds to A80569 $(r = 3)$, A87977 $(r = 4)$, A87978 $(r = 5)$, A138206 $(r = 6)$, A138207 $(r = 7)$, or A154573 $(r = 8)$. However, note that these OEIS sequences list the starter of the first run of size *at least* s rather than size *exactly* s, so they begin with N_r (for $s = 1$) and have repeated terms wherever the sequence $(n(r, s) : s \geq 2)$ fails to increase monotonically. The instances with $n \leq 10^{12}$ where this occurs are

$$n(3, 12) = 534\,078, \quad n(3, 13) = 2\,699\,915, \quad n(3, 14) = 526\,095;$$
$$n(4, 6) = 2\,713\,332, \quad n(4, 7) = 1\,217\,250;$$
$$n(4, 18) = 203\,594\,236\,366, \quad n(4, 19) = 118\,968\,284\,928.$$

To accommodate such failures of monotonicity, we have introduced [4] the OEIS sequences A185032 and A185042. For fixed size s, the sequence $(n(r, s) : r \geq 1)$ similarly corresponds to sequence A93548 $(s = 2)$ or A93549 $(s = 3)$. The OEIS commentaries credit Fuller [6] with $n(8, 2)$ and $n(9, 2)$, while Johnson [8, 9] is credited with $n(10, 2)$, $n(11, 2)$ and also $n(7, 3)$ and $n(8, 3)$.

The commentaries on the OEIS sequences for constant rank r indicate that Johnson [10] computed $\omega(n \leq 10^{13})$. In the interval $n \leq 10^{12}$ our computations independently confirm his announced results. For convenience, we now assemble the record holders for $n \leq 10^{12}$ together with Fuller's and Johnson's additional contributions:

$$\begin{aligned}
r = 3:\ & n(3, 18) = 146\,216\,247\,221 < 10^{12}, \text{ and} \\
& n(3, s) > 10^{13} \text{ for } s \geq 19, \text{ if it exists.} \\
r = 4:\ & n(4, 23) = 585\,927\,201\,062 < 10^{12}, \text{ and} \\
& n(4, s) > 10^{13} \text{ for } s = 20, 21, 22 \text{ and } s \geq 24, \text{ if they exist.} \\
r = 5:\ & n(5, 10) = 287\,980\,277\,114 < 10^{12}, \\
& n(5, 11) = 1\,182\,325\,618\,032 > 10^{12},
\end{aligned}$$

$$
\begin{aligned}
& n(5,12) = 6\,455\,097\,761\,454 > 10^{12}, \text{ and} \\
& n(5,s) > 10^{13} \text{ for } s \geq 13, \text{ if it exists.} \\
r = 6:\ & n(6,6) = 626\,804\,494\,291 < 10^{12}, \\
& n(6,7) = 7\,563\,009\,743\,844 > 10^{12}, \text{ and} \\
& n(6,s) > 10^{13} \text{ for } s \geq 8, \text{ if it exists.} \\
r = 7:\ & n(7,3) = 30\,989\,984\,674 < 10^{12}, \\
& n(7,4) = 1\,673\,602\,584\,618 > 10^{12}, \text{ and} \\
& n(7,s) > 10^{13} \text{ for } s \geq 5, \text{ if it exists.} \\
r = 8:\ & n(8,2) = 65\,893\,166\,030 < 10^{12}, \\
& n(8,3) = 10\,042\,712\,381\,260 \approx 10^{13}, \text{ and} \\
& n(8,s) > 10^{13} \text{ for } s \geq 4, \text{ if it exists.} \\
r = 9:\ & n(9,2) = 5\,702\,759\,516\,090 < 10^{13}, \text{ and} \\
& n(9,s) > 10^{13} \text{ for } s \geq 3, \text{ if it exists.} \\
& n(r,s) > 10^{13} \text{ for } r \geq 10 \text{ and } s \geq 2, \text{ if it exists.} \\
r = 10:\ & n(10,2) = 490\,005\,293\,940\,084 < 10^{15}. \\
r = 11:\ & n(11,2) = 76\,622\,240\,600\,506\,314 < 10^{17}.
\end{aligned}
$$

These results do not improve the lower bounds $s(P_3) \geq 18$ and $s(P_4) \geq 23$, but do give us the improved bounds $s(P_5) \geq 12$, $s(P_6) \geq 7$, $s(P_7) \geq 4$, and $s(P_8) \geq 3$.

In [5] we remarked on failures of monotonicity, in the sequences $(n(r,s) : s \geq 2)$ for fixed r. Now we can note the more striking example

$$n(4,19) < n(3,18)$$

showing the failure of monotonicity in both parameters simultaneously.

3. Higher Ranks

Partition the first st primes $\{p_1, p_2, \cdots, p_{st}\}$ into s sets of size t, with products m_i for $0 \leq i < s$, so $m_0 m_1 \cdots m_{s-1} = N_{st}$. By the Chinese Remainder Theorem, there is a unique positive integer $n_0 < N_{st}$ such that

$$n_0 + i \equiv 0 \pmod{m_i} \text{ for } 0 \leq i < s.$$

Let $n_k = n_0 + kN_{st}$, for $k \geq 0$. There are integers q_i such that $n_0 + i = m_i q_i$ for $0 \leq i < s$, so we can define $q_{i,k} = q_i + kN_{st}/m_i$ for $0 \leq i < s$ and

$k \geq 0$. Then $n_k + i = m_i q_{i,k}$ for $0 \leq i < s$, $k \geq 0$. It follows that $\omega(n_k + i) \geq \omega(m_i) = t$ for $0 \leq i < s$.

There are heuristic grounds to expect that $n_k^{[s]} \subset P_r$ for some relatively small values of k and various ranks r equal to or slightly larger than t. Each "success" provides us with an instance of a plateau of rank $r \geq t$ and size at least s. We call this the *Chinese Remainder method* for selecting likely starters of principal rank plateaux. For our main implementation we chose s products m_i of comparable magnitude by partitioning the first st primes so that

$$m_i = \prod \{p_j : 1 \leq j \leq st \mid j \equiv i+1, -i \pmod{2s}\} \text{ for } 0 \leq i < s.$$

This produced instances of plateaux of rank r and size s for $9 \leq r \leq 64$ with $s = 2$, for $8 \leq r \leq 32$ with $s = 3$, for $7 \leq r \leq 19$ with $s = 4$, for $7 \leq r \leq 15$ with $s = 5$, and for $7 \leq r \leq 11$ with $s = 6$ (see [3]). Of course, the starters in all cases are larger than 10^{13}; indeed, most are very much larger. In particular, the instance with $(r, s) = (53, 2)$ arises from $t = 50$, $k = 4$ and the 221 digit starter

$$\begin{aligned} n_4 = {} & 19\,749\,679\,144\ \ 729\,629\,383\,908\,714\ \ 098\,269\,221\,603\,094 \\ & 436\,889\,433\,057\,615\ \ 898\,886\,873\,725\,626\ \ 596\,853\,647\,892\,257 \\ & 452\,228\,874\,060\,200\ \ 487\,940\,193\,303\,962\ \ 861\,555\,803\,779\,979 \\ & 497\,010\,248\,369\,524\ \ 239\,457\,369\,352\,851\ \ 534\,301\,395\,043\,448 \\ & 026\,143\,509\,096\,224\ \ 791\,036\,513\,898\,994\ \ 716\,984\,922\,582\,034. \end{aligned}$$

4. Current Best Lower Bounds

Combining our Chinese Remainder method results with the earlier data from our computations and Johnson's, we have

Theorem 4.1. *The following are lower bounds for the maximum length $s(P_r)$ of any run of consecutive integers with principal rank r:*

$$s(P_3) \geq 18,\ s(P_4) \geq 23,\ s(P_5) \geq 12,\ s(P_6) \geq 10,\ s(P_7) \geq 8,\ s(P_8) \geq 7;$$
$$s(P_r) \geq 6 \textit{ for } r \in \{9, 10, 11\},\ s(P_r) \geq 5 \textit{ for } r \in \{12, \ldots, 15\},$$
$$s(P_r) \geq 4 \textit{ for } r \in \{16, \ldots, 19\},\ s(P_r) \geq 3 \textit{ for } r \in \{20, \ldots, 32\},$$
$$\textit{and } s(P_r) \geq 2 \textit{ for } r \in \{33, \ldots, 64\}.$$

The sporadic cases $s(P_{23}) \geq 4$, $s(P_{67}) \geq 2$ and $s(P_{69}) \geq 2$ were found fortuitously.

Based on the starters we computed by the Chinese Remainder method, we can specify simple upper bounds on the starters of the long runs implied by Theorem 4.1.

Theorem 4.2. *The sequence $\omega(n \leq 10^a)$ contains at least one plateau of rank r and size s in the following cases:*

$a = \ \ 50:$ $(r,s) = (6,10),(7,8),(8,7),(9,6)$.
$a = 100:$ $(r,s) = (10,6),(11,6),(12,5),(13,5),(14,5),(16,4),(17,4),(20,3)$.
$a = 150:$ $(r,s) = (15,5),(18,4),(19,4),(21,3),(22,3),(23,4),(24,3),(25,3),$ $(26,3),(27,3),(28,3),(29,3),(33,2),(34,2),(35,2),(36,2),$ $(37,2),(38,2),(39,2)$.
$a = 200:$ $(r,s) = (30,3),(31,3),(32,3),(40,2),(41,2),(42,2),(43,2),(44,2),$ $(45,2),(46,2),(47,2),(48,2),(49,2),(50,2)$.
$a = 250:$ $(r,s) = (51,2),(52,2),(53,2),(54,2),(55,2),(56,2),(57,2),(58,2),$ $(59,2)$.
$a = 300:$ $(r,s) = (60,2),(61,2),(62,2),(63,2),(64,2)$.
$a = 350:$ $(r,s) = (67,2),(69,2)$.

The interested reader is referred to the reports [2, 3] for discussion and further details of the data supporting this paper. In particular, more detail on Theorem 4.2 is given in the Addendum to [3]: for example, there is a plateau of rank 6 and size 10 in the interval $44.90 < a < 44.91$.

In view of the success of the Chinese Remainder method, with no apparent limitations in principle for higher ranks, it is natural to propose (see [5])

Conjecture 4.3. $s(P_r) \geq 2$ *for all* $r \geq 1$.

5. Asymptotic Behaviour

If P_r contains infinitely many runs of size s^*, but only finitely many larger runs, we denote s^* by $s^*(P_r)$. Thus

$$s^*(P_r) = \limsup_N \{s : n^{[s]} \subset P_r,\ n > N\}.$$

Trivially $s^*(P_r) \geq 1$, but considerably stronger than Conjecture 4.3 is

Conjecture 5.1. $s^*(P_r) \geq 2$ *for all* $r \geq 1$.

In particular, we know $s^*(P_1) \leq 2$, and $s^*(P_1) = 2$ holds just if there are infinitely many Fermat or Mersenne primes. Utilizing recent sieve results, we can in fact prove Conjecture 4.3 completely, and Conjecture 5.1 for $r \geq 3$, as follows.

A set of $k \geq 2$ linear forms $S(x) = \{L_i(x) = a_i x + b_i : 1 \leq i \leq k\}$ is *admissible* if for each prime p there is an integer n_p such that none of the integers in $S(n_p)$ is a multiple of p. The key theorem we shall apply is

Theorem 5.2 (Goldston, Graham, Pintz and Yıldırım [7])**.** *If* $S(x) = \{L_i(x) = a_i x + b_i : 1 \leq i \leq 3\}$ *is an admissible set of three linear forms,* *then for infinitely many integers n at least two of the integers in $S(n)$ are products of exactly two distinct primes, each of which is greater than $n^{1/144}$.*

We are greatly indebted to an anonymous referee for sharing with us the $r = 3$ argument in the following theorem.

Theorem 5.3. *$s^*(P_3) \geq 2$ and $s^*(P_4) \geq 2$.*

Proof. The set $S_3(x) = \{4x + 3, 6x + 5, 9x + 7\}$ is admissible, verified by evaluating at $x_2 = 0$ and $x_p = p - 1$ for $p \geq 3$. Now suppose $n > 3^{144}$ is such that at least two integers in $S_3(n)$ are products of two distinct primes, each greater than 3. Then one of the sets

$$\{3(4n + 3), 2(6n + 5)\},\ \{9(4n + 3), 4(9n + 7)\},\ \{2(9n + 7), 3(6n + 5)\}$$

is a pair of consecutive integers in P_3. It follows from Theorem 5.2 that there are infinitely many such pairs, so $s^*(P_3) \geq 2$.

Similarly, $S_4(x) = \{200x + 161, 210x + 169, 441x + 355\}$ is admissible, so for infinitely many integers $n > 7^{144}$ at least two integers in $S_4(n)$ are products of two distinct primes, each greater than 7. For any such n, one of the sets

$$\{20(210n + 169), 21(200n + 161)\},$$
$$\{200(441n + 355), 441(200n + 161)\},$$
$$\{21(210n + 169), 10(441n + 355)\}$$

is a pair of consecutive integers in P_4. Hence $s^*(P_4) \geq 2$. □

Adapting the ideas used in proving Theorem 5.3, we shall now show

Lemma 5.4. *For any $r \geq 1$, if $s(P_r) \geq 2$ and some pair in P_r includes a multiple of 3, then $s^*(P_{r+2}) \geq 2$, and infinitely many pairs in P_{r+2} include a multiple of 3.*

Proof. Suppose $\{A, A + 1\} \subset P_r$ satisfies $A(A + 1) \equiv 0 \pmod 3$. Let $S(x)$ comprise the three linear forms

$$L_1(x) = A^2x + (A - 1)^2,$$
$$L_2(x) = A(A + 1)x + A^2 - A - 1,$$
$$L_3(x) = (A + 1)^2x + A^2 - 2.$$

Each linear form has coprime coefficients. The mod 3 condition ensures that no member of $S(1)$ is divisible by 2 or 3. For any prime $p \geq 5$ there are at most three residue class solutions to

$$L_1(x)\, L_2(x)\, L_3(x) \equiv 0 \pmod p,$$

so there are residue classes $n \pmod p$ for which no member of $S(n)$ is a multiple of p. Hence $S(x)$ is an admissible set.

If q is the largest prime factor of $A(A+1)$, Theorem 5.2 ensures that there are infinitely many integers $n > q^{144}$ such that at least two integers in $S(n)$ are products of exactly two distinct primes, each greater than q. For each such n, at least one of the sets

$$\{AL_2(n),\ (A+1)L_1(n)\},$$
$$\{(A+1)L_2(n),\ AL_3(n)\},$$
$$\{A^2L_3(n),\ (A+1)^2L_1(n)\}$$

is a pair of consecutive integers in P_{r+2}. Since each set contains a multiple of 3, the lemma follows. □

The set $S(x)$ used in the proof of Lemma 5.4 is not necessarily the only suitable choice. For example, neither of the sets $S_3(x)$ and $S_4(x)$ used for Theorem 5.3 exactly conforms to $S(x)$.

Theorem 5.5. *$s^*(P_r) \geq 2$, and infinitely many pairs in P_r include a multiple of 3, for all $r \geq 3$.*

Proof. If $s^*(P_r) \geq 2$, and infinitely many pairs in P_r include a multiple of 3, any such pair ensures that $s^*(P_{r+2}) \geq 2$, and infinitely many pairs in P_{r+2} include a multiple of 3, by Lemma 5.4. The linear forms constructed in Theorem 5.3 establish the instances $r = 3$ and $r = 4$, so iteration yields the theorem. □

Since $s(P_1) = 4$, $s(P_2) \geq 8$, and $s(P_r) \geq s^*(P_r)$, Conjecture 4.3 is now completely proved, and Conjecture 5.1 is proved except in the key instances $r = 1$ and $r = 2$.

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